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A Singular Multi-Parameter Eigenvalue Problem in Second Order Ordinary Differential Equations

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1. INTRODUCTION

In a recent paper [1] I considered, from an abstract point of view, a multiparameter eigenvalue problem, the results of which had applications to systems of ordinary differential equations of the type discussed by Faierman [2]. More specifically, the system has the form

$$\frac{d^2y_r(x_r)}{dx_r^2} + q_r(x_r)\, y_r(x_r) + \sum_{s=1}^{k} \lambda_s a_{rs}(x_r)\, y_r(x_r) = 0, \tag{1}$$

$a_r \leqslant x_r \leqslant b_r$, $q_r \in C([a_r, b_r])$ and real valued, $a_{rs} \in C([a_r, b_r])$ and real valued, and λ_s complex parameters, subject to the definiteness condition

$$\det\{a_{rs}(x_r)\} > 0$$

and boundary conditions

$$y_r(a_r)\cos\alpha_r + y_r'(a_r)\sin\alpha_r = 0, \qquad 0 < \alpha_r \leqslant \pi, \tag{2}$$

$$y_r(b_r)\cos\beta_r + y_r'(b_r)\sin\beta_r = 0, \qquad 0 \leqslant \beta_r < \pi, \quad r, s = 1, 2,..., k. \tag{3}$$

The analysis of this system as presented in these two papers depends essentially on the finiteness of the intervals $[a_r, b_r]$. The aim of this paper is to investigate the system when some (or all) of the variables x_r are allowed to range over a half line $[a_r, \infty)$.

2. FUNCTIONS OF BOUNDED VARIATION

We give here results concerning real valued functions of bounded variation in k real variables which will be needed in later considerations.

For $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\beta = (\beta_1, \ldots, \beta_k) \in R^k$ with $\beta_r > \alpha_r$, $r = 1, \ldots, k$, $[\alpha, \beta]$ will denote the interval in R^k given by $\{(x_1, \ldots, x_k) \mid \alpha_r \leqslant x_r \leqslant \beta_r, r = 1, \ldots, k\}$ while $I = [c, d]$ will denote any subinterval of $[\alpha, \beta]$ with "sides" parallel to those of $[\alpha, \beta]$. A subdivision σ of $[\alpha, \beta]$ consists of a finite number of intervals having at most sides and vertices in common and whose union is $[\alpha, \beta]$. Let $f(x_1, \ldots, x_k)$ be a real-valued function defined on $[\alpha, \beta]$. Then for $I = [c, d]$ we define

$$\Delta_r f(x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_k) = f(x_1, \ldots, x_{r-1}, d_r, x_{r+1}, \ldots, x_k) - f(x_1, \ldots, x_{r-1}, c_r, x_{r+1}, \ldots, x_k)$$

for $r = 1, \ldots, k$ and

$$\Delta_f(I) = \Delta_1 \Delta_2 \cdots \Delta_k f.$$

The *function* f *is said to be bounded variation on* $[\alpha, \beta]$ *if* $\sum_{I \in \sigma} | \Delta_f(I)|$ is bounded with respect to subdivisions σ of $[\alpha, \beta]$. The total variation f over $[\alpha, \beta]$ is given by

$$V(f, [\alpha, \beta]) = \sup_{\sigma} \sum_{I \in \sigma} | \Delta_f(I)|.$$

The function f is said to be positively monotonic on $[\alpha, \beta]$ if $\Delta_f(I) \geqslant 0$ for all subintervals $I \subseteq [\alpha, \beta]$ and negatively monotonic if $\Delta_f(I) \leqslant 0$ for all such I. We see from [4, Sections 46.3, 46.5; p. 249] that if f is positively monotonic it is of bounded variation and $V(f, [\alpha, \beta]) = \Delta_f([\alpha, \beta])$, while if f is negatively monotonic it is also of bounded variation and $V(f, [\alpha, \beta]) = -\Delta_f([\alpha, \beta])$.

LEMMA 2.1. *Let* f *be positively monotonic on* $[\alpha, \beta] \subset R^k$ *and suppose* $f(x, \ldots, x_k)$ *is zero if* $\exists i$ *such that* $x_i = \alpha_i$. *Then if* $x_{i_1}, \ldots, x_{i_p}$, $1 \leqslant p \leqslant k$ *are fixed at the values* $x'_{i_1}, \ldots, x'_{i_p}$ *respectively,* f *is positively monotonic in the remaining variables.*

Proof. Without loss of generality we may assume $x_1, \ldots, x_p$ to be fixed at $x_1', \ldots, x_p'$ and that each $x_i' \neq \alpha_i$ for otherwise f would be zero. Let J be a subinterval of $[\alpha_{p+1}, \beta_{p+1}] \times \cdots \times [\alpha_k, \beta_k]$. We are required to show that, with respect to J, $\Delta_{p+1} \cdots \Delta_k f(x_1', \ldots, x_p') \geqslant 0$.

Let $I = [\alpha_1, x_1'] \times \cdots \times [\alpha_p, x_p'] \times J$. Then since f is positively monotonic on $[\alpha, \beta]$, we see that with respect to I, $\Delta_1 \cdots \Delta_k f \geqslant 0$. Now $\Delta_1 \Delta_2 \cdots \Delta_k f = \Delta_{p+1} \cdots \Delta_k \Delta_1 \cdots \Delta_p f$ since the operators Δ_r commute. Further $\Delta_p f(x_1 \cdots x_{p-1}, x_{p+1}, \ldots, x_k) = f(x_1, \ldots, x_{p-1}, x_p', x_{p+1}, \ldots, x_k)$ since

$$f(x_1, \ldots, x_{p-1}, \alpha_p, x_{p+1}, \ldots, x_k) = 0.$$

Continuing thus, we have

$$0 \leqslant \Delta_1 \cdots \Delta_k f = \Delta_{p+1} \cdots \Delta_k f(x_1', \ldots, x_p')$$

which is the desired result.

We note here that with suitable modifications, the result holds true if f is negatively monotonic or if there is some other vertex γ of $[\alpha, \beta]$ such that $f(x_1, ..., x_k) = 0$ if $\exists i$ such that $x_i = \gamma_i$. In such cases, it may be necessary to change the conclusion of the lemma to read, "f is negatively monotonic in the remaining variables," depending on the original nature of f and the position of γ relative to α. However, the proofs are similar to that given above.

LEMMA 2.2. *Let f be positively monotonic on $[\alpha, \beta] \subset R^k$ and suppose $f(x_1, ..., x_k)$ is zero if $\exists i$ such that $x_i = \alpha_i$. Then for each $x \in [\alpha, \beta]$ all the limits $f(x_1 \pm 0, ..., x_k \pm 0)$ (where meaningful) exist.*

Proof. We argue by induction on the dimension k. When $k = 1$ the lemma is in effect the well known result that the limits $f(x \pm 0)$ exist for a nondecreasing function f of a real variable x. Suppose now that the result holds for dimensions $1, 2, ..., k - 1$ and let $x' \in [\alpha, \beta)$. We shall show the existence of $f(x_1' + 0, ..., x_k' + 0)$—the other cases are similar. Consider intervals of the type $[x', x]$. Since Δf is an additive function of intervals [4, Sections 45.4, p. 246] and nonnegative, $\lim_{x \to x'} \Delta f([x', x])$ exists. Further, $\Delta_f([x', x]) = f(x_1, ..., x_k) +$ terms where f is evaluated at other vertices of $[x', x]$. These other vertices of $[x', x]$ have at least one coordinate fixed at some x_i'. For example, if we consider the vertex $(x_1', x_2, ..., x_k)$, the previous lemma and our inductive hypothesis show that $\lim_{x \to x'} f(x_1', x_2, ..., x_k)$ exists. Similarly the limits of the other terms exist and so $\lim_{x \to x'} f(x)$ also exists, i.e., $f(x_1' + 0, ..., x_k' + 0)$ exists.

Again we note that this result holds true if f is negatively monotonic or if there is some other vertex γ of $[\alpha, \beta]$ such that $f(x_1, ..., x_k) = 0$ if $\exists i$ such that $x_i = \gamma_i$.

We define the oscillation of f at the point $x \in [\alpha, \beta]$ by

$$w(f, x) = \inf_{\substack{I \subseteq [\alpha, \beta] \\ x \in I}} \sup_{y', y \in I} |f(y) - f(y')|.$$

Here, as usual, I denotes a subinterval of $[\alpha, \beta]$.

LEMMA 2.3. *Let f be as in the previous two lemmas. Then the discontinuities of f lie on a countable collection of $(k - 1)$ dimensional hyperplanes parallel to the coordinate axes.*

Proof. Since $f(x)$ has limits for each $x \in [\alpha, \beta]$ provided approach to x is restricted to orthands, it follows from the Cauchy condition for convergence that for each $x \in [\alpha, \beta]$ and for a given $\epsilon > 0$ there exists a k-dimensional sphere with x as center such that if x' is interior to this sphere but not on the $(k-1)$-dimensional hyperplanes through x parallel to the coordinate axes, then $w(f, x') < \epsilon$. $[\alpha, \beta]$ being compact, a finite number of these spheres cover it. If the centers of these covering spheres are $x^{(p)}$, $p = 1, 2, \ldots, P$, it follows that if x' is not on any of the $(k-1)$-dimensional hyperplanes through the $x^{(p)}$ and parallel to the coordinate axes, then $w(f, x') < \epsilon$. Thus the points of $[\alpha, \beta]$ for which $w(f, x) \geqslant \epsilon$ lie on this finite collection of $(k-1)$-dimensional hyperplanes. By setting $\epsilon = 1/m$, we obtain a method for counting the hyperplanes on which the discontinuities of f lie.

THEOREM 2.1 (Helly Theorem). *Let $f_n(x)$, $n = 1, 2, \ldots$ be a sequence of positively monotonic functions on $[\alpha, \beta]$ such that $f_n(x_1, \ldots, x_k)$ is zero if $\exists i$ such that $x_i = \alpha_i$. Suppose there is a positive constant K such that $V(f_n, [\alpha, \beta]) \leqslant K$ for each n. Then there is a subsequence f_{n_m} of f_n such that $\lim_{m\to\infty} f_{n_m}(x)$ exists for all $x \in [\alpha, \beta]$. The limit function f is positively monotonic and $f(x_1, \ldots, x_k)$ is zero if $\exists i$ such that $x_i = \alpha_i$.*

Proof. We proceed by induction on the dimension k. In the case $k = 1$, our theorem is a special case of the usual Helly theorem for sequences of nondecreasing functions of a real variable, (see [3] p. 44). Suppose now that the result holds for dimensions $1, 2, \ldots, k-1$.

Let $Z = \{x^1, x^2, \ldots\}$ be a countable subset of $[\alpha, \beta]$ with the properties

(i) Z is dense in $[\alpha, \beta]$,

(ii) if E is any p-dimensional face of $[\alpha, \beta]$, $0 \leqslant p \leqslant k-1$, then $E \cap Z$ is dense in E in the relative topology on E.

From the properties of the functions f_n we have

$$0 \leqslant f_n(x) = \Delta_{f_n}([\alpha, x]) \leqslant \Delta_{f_n}([\alpha, \beta]) \leqslant K, \qquad n = 1, 2, \ldots .$$

Thus the double sequence $f_n(x^q)$, $x^q \in Z$, satisfies the conditions of Helly's Selection Theorem [3, p. 44], and so there is a subsequence such that $f(x) = \lim_{m\to\infty} f_{n_m}(x)$ exists for all $x \in Z$. If $(x_1, \ldots, x_k)$ is such that $\exists i$ with $x_i = \alpha_i$, we may define $f(x) = 0$ (for in this case each $f_n(x) = 0$). Thus so far we have a subsequence f_{n_m} converging to a limit f on Z' where

$$Z' = Z \cup \{(x_1, \ldots, x_k) \mid \exists i \text{ with } x_i = \alpha_i\}.$$

It is clear that if we consider subintervals I of $[\alpha, \beta]$ whose vertices are in Z', then $\Delta_f(I) \geqslant 0$. An argument similar to that of Lemma 2.1 shows that if $x_{i_1}, \ldots, x_{i_p}$ are fixed at $x'_{i_1}, \ldots, x'_{i_p}$ respectively, $1 \leqslant p \leqslant k$, then $f(x)$ is positively monotonic in the remaining variables (subject of course to $x \in Z'$).

If y is any point in $[\alpha, \beta]$ then all the limits of $f(x)$ as x approaches $(y_1 \pm 0, \ldots, y_k \pm 0)$ through the set Z' exist. The proof is similar to that of Lemma 2.2 and we sketch its details. When $k = 1$, the result is trivial. Suppose the result holds for dimensions $1, 2, \ldots, k-1$. Let $x \in [\alpha, \beta)$. We shall show that as y approaches $(x_1 + 0, \ldots, x_k + 0)$ through Z', $f(y)$ has a limit. Select $z \in Z'$ such that $z_i \leqslant x_i$, $i = 1, 2, \ldots, k$. That such a z exists follows readily from the properties of Z'. We now consider intervals of the type $[z, y]$ and proceed as in the proof of Lemma 2.2.

The proof of Lemma 2.3 shows that these limits through orthands will coincide for all $x \in [\alpha, \beta]$ with the exception of a collection of points lying on a countable family $H_1, H_2, \ldots$ of $(k-1)$-dimensional hyperplanes parallel to the coordinate axes. Thus we may now extend f to $[\alpha, \beta] - \bigcup_{i=1}^{\infty} H_i$. Let x be a point in this set. We need to show $\lim_{m \to \infty} f_{n_m}(x) = f(x)$. If $x \in Z'$, there is nothing to prove. If $x \notin Z'$, for each $q = 1, 2, \ldots$ let $z^q, y^q \in Z'$ be such that $z_i^q \geqslant x_i$, $z_i^q \downarrow x_i$, $y_i^q \leqslant x_i$ and $y_i^q \uparrow x_i$, $i = 1, 2, \ldots, k$. Again the properties of Z' ensure the existence of the points z^q, y^q. Then since $[\alpha, y^q] \subseteq [\alpha, x] \subseteq [\alpha, z^q]$ we have

$$\Delta f_{n_m}([\alpha, y^q]) \leqslant \Delta f_{n_m}([\alpha, x]) \leqslant \Delta f_{n_m}([\alpha, z^q]),$$

i.e.,

$$f_{n_m}(y^q) \leqslant f_{n_m}(x) \leqslant f_{n_m}(z^q)$$

for each m and q. Letting $m \to \infty$ gives

$$f(y^q) \leqslant \liminf_{m \to \infty} f_{n_m}(x) \leqslant \limsup_{m \to \infty} f_{n_m}(x) \leqslant f(z^q).$$

Note that f is continuous at x, so taking the limit as $q \to \infty$ gives

$$f(x) \leqslant \liminf_{m \to \infty} f_{n_m}(x) \leqslant \limsup_{m \to \infty} f_{n_m}(x) \leqslant f(x)$$

and hence

$$f(x) = \lim_{m \to \infty} f_{n_m}(x).$$

We now have $f_{n_m} \to f$ on $[\alpha, \beta] - \bigcup_{i=1}^{\infty} H_i$. On H_i, we see from our lemmas and the inductive hypothesis that there is a subsequence $f_{n_{1m}}$ of f_{n_m} converging to a limit (say f) on H_1. By the same argument, there is a subsequence $f_{n_{2m}}$ of $f_{n_{1m}}$ converging to a limit (say f) on H_1 and H_2. Continuing in this way we see that the sequence $f_{n_{mm}}$—a subsequence of the sequence f_n—converges to a limit, f, for all $x \in [\alpha, \beta]$. It is obvious that the limit function has the properties described in the last part of the theorem. This completes the proof.

We note that as with the previous results this theorem remains valid if there is some other vertex γ of $[\alpha, \beta]$ such that each $f_n(x_1, \dots, x_k) = 0$ if $\exists i$ with $x_i = \gamma_i$.

A function $f(x)$ is said to be left continuous if for each $x = (x_1, \dots, x_k)$

$$\lim_{\epsilon_1 \to 0+, \dots, \epsilon_k \to 0+} f(x_1 - \epsilon_1, \dots, x_k - \epsilon_k) = f(x_1, \dots, x_k).$$

If f is positively monotonic on $[\alpha, \beta]$ it is possible to develop a theory of integration with respect to the additive function of intervals $\Delta_f(I)$. We write such integrals as $\int_{[\alpha,\beta]} g(x)\, df(x)$. This theory is discussed in detail in [4, Chap. VII, pp. 242–311]. When f is left continuous, the measure of the half open internal $[c, d)$—obtained by evaluating $\int_{[\alpha,\beta]} \chi_{[c,d)}(x)\, df(x)$—coincides with $\Delta_f([c, d))$ [4, Section 47.5, p. 254].

Finally in this section we state

THEOREM 2.2. *Let f_n be a sequence of positively monotonic functions on $[\alpha, \beta]$ with pointwise limit the positively monotonic function f. Then if $g(x)$ is continuous on $[\alpha, \beta]$,*

$$\lim_{n\to\infty} \int_{[\alpha,\beta]} g(x)\, df_n(x) = \int_{[\alpha,\beta]} g(x)\, df(x)$$

Proof. Note that the sequence $V(f_n, [\alpha, \beta]) = \Delta_{f_n}([\alpha, \beta])$ is bounded, being convergent to $\Delta_f([\alpha, \beta])$. Further, for any subinterval $I \subseteq [\alpha, \beta]$, $\Delta_{f_n}(I) \to \Delta_f(I)$. The result now follows from [5, Section 4.5, Theorem 3, p. 72].

3. KNOWN RESULTS

We now restate the results of [1 and 2] in a form more suited to our present purposes.

An eigenvalue $\lambda = (\lambda_1, \dots, \lambda_k)$ and eigenfunction $\prod_{r=1}^k f_r(x_r)$ for the problem (1)–(3) is a k-tuple of complex numbers and functions $f_r(x_r)$, $r = 1, 2, \dots, k$ satisfying the system (1) and boundary conditions (2), (3). It has been shown in [1 and 2] that the eigenvalues consist of real k-tuples and that the eigenfunctions are complete in $L^2(I_k)$ with respect to the weight function $|A|(x) = \det\{a_{rs}(x_r)\}$; $-I_k$ is the Cartesian product of the intervals $[a_r, b_r]$, $r = 1, \dots, k$. Note that in [2] certain differentiability conditions were placed on the functions q_r, a_{rs}, $r, s = 1, \dots, k$ but that in [1] these were relaxed to the continuity conditions stated in (1).

Let $\varphi_r(x_r, \lambda_1, \dots, \lambda_k)$, $r = 1, \dots, k$ be the solutions of the system (1) satisfying

$$\varphi_r(a_r) = \sin \alpha_r, \qquad \varphi_r'(a_r) = -\cos \alpha_r, \qquad r = 1, 2, \dots, k,$$

for all $(\lambda_1, \ldots, \lambda_k)$. We shall put $\varphi(x, \lambda) = \prod_{r=1}^{k} \varphi_r(x_r, \lambda_1, \ldots, \lambda_k)$. Then if $\prod_{r=1}^{k} \psi_r(x_r, \lambda^m)$ is the $|A|(x)$-normalized eigenfunction corresponding to the eigenvalue λ^m, there exists a constant $\gamma(\lambda^m) \neq 0$ such that

$$\prod_{r=1}^{k} \psi_r(x_r, \lambda^m) = \gamma(\lambda^m)\, \varphi(x, \lambda^m).$$

Now the completeness result states that for $f(x) \in L^2(I_k)$

$$\begin{aligned}\int_{I_k} |A|(x)\, |f(x)|^2\, dx &= \sum_m \left| \int_{I_k} |A|(x) f(x) \prod_{r=1}^{k} \psi_r(x_r, \lambda^m)\, dx \right|^2 \\ &= \sum_m |\gamma(\lambda^m)|^2 \left| \int_{I_k} |A|(x) f(x)\, \varphi(x, \lambda^m)\, dx \right|^2.\end{aligned}$$

For an interval $J \subset R^k$ whose closure is $[c, d]$ we define $\Omega(J) = \sum_m |\gamma(\lambda^m)|^2$ where the summation is extended over those indices m for which $\lambda^m \in [c, d)$. The number of terms in this summation is finite since it was shown in [1] that the eigenvalues have no finite point of accumulation. Then Ω is a non-negative function of intervals and so by [4, Section 45.5, p. 246] there is a positively monotonic function $\rho(\lambda)$ defined on R^k such that for any interval $J \subset R^k$, $\Omega(J) = \Delta_\rho(J)$. We also have that ρ is left continuous and $\rho(\lambda) = 0$ if any of the numbers $\lambda_1, \ldots, \lambda_k$ is zero. It is clear that integration of a function f with respect to ρ is equivalent to summation of the sequence with respect to the weight sequence $|\gamma(\lambda^m)|^2$.

For a given $f(x) \in L^2(I_k)$ we define

$$(Uf)(t) = \int_{I_k} |A|(x) f(x)\, \varphi(x, t)\, dx.$$

Then we may now write the Parseval equality as

$$\int_{I_k} |A|(x)\, |f(x)|^2\, dx = \int_{R^k} |(Uf)(t)|^2\, d\rho_b(t).$$

We have rewritten the function ρ as ρ_b to display the fact that this analysis is relevant to $I_k = \times_{r=1}^{k} [a_r, b_r]$. Our problem is to keep the point $a = (a_1, \ldots, a_k)$ fixed and vary $b = (b_1, \ldots, b_k)$ by letting some (or all) of the b_r tend to infinity.

We note in passing here, that a standard argument shows that the spaces $L^2(I_k, \gamma)$ where for $E \subseteq I_k$, $\gamma(E) = \int_E |A|(x)\, dx$, and $L^2(\rho_b)$ are unitarily equivalent under the mapping $f \to Uf$ described above.

4. A Limiting Spectral Function

LEMMA 4.1. *Let $T > 0$ be given. Then there exists $a_r' > a_r$, $r = 1,..., k$ and $\epsilon > 0$ such that*

$$\int_{[a,a']} | A |(x)\, \varphi(x, t)\, dx \geqslant \epsilon$$

for $| t_r | \leqslant T$, $r = 1, 2,..., k$.

Proof. The result is immediate in the case that $\varphi(a, t) = \prod_{r=1}^{k} \sin \alpha_r \neq 0$ for then, $\varphi(x, t)$ being continuous in all variables, we may choose $a_r' > a_r$ so that $\varphi(x, t) \geqslant \frac{1}{2} \prod_{r=1}^{k} \sin \alpha_r$ for $a_r \leqslant x_r \leqslant a_r'$ and $| t_r | \leqslant T$, $r = 1,..., k$. We have then

$$\int_{[a,a']} | A |(x)\, \varphi(x, t)\, dx \geqslant \frac{1}{2} \prod_{r=1}^{k} \sin \alpha_r \int_{[a,a']} | A |(x)\, dx > 0.$$

The case $\prod_{r=1}^{k} \sin \alpha_r = 0$ can occur by some (or all) of the α_r having the value π (recall $\alpha_r \in (0, \pi]$). Let us suppose, for the sake of definiteness that $\alpha_1 = \pi$ and $\alpha_r \neq \pi$, $r = 2, 3,..., k$. Then we have $\varphi_1'(a_1, t) = 1$ for all t. Thus we may select $a_1' > a_1$ so that $\varphi_1'(x_1, t) \geqslant \frac{1}{2}$ for $a_1 \leqslant x_1 \leqslant a_1'$, $| t_r | \leqslant T$. Then $\varphi_1(x_1, t) \geqslant \frac{1}{2}(x_1 - a_1)$, $a_1 \leqslant x_1 \leqslant a_1'$, $| t_r | \leqslant T$. As before, we choose $a_2',..., a_k'$ so that

$$\prod_{r=2}^{k} \varphi_r(x_r, t) \geqslant \frac{1}{2} \prod_{r=2}^{k} \sin \alpha_r \qquad \text{for} \quad a_r \leqslant x_r \leqslant a_r', \quad r = 2,..., k,$$

and $| t_r | \leqslant T$, $r = 1,..., k$. Then we have

$$\int_{[a,a']} | A |(x)\, \varphi(x, t)\, dx \geqslant \frac{1}{4} \prod_{r=2}^{k} \sin \alpha_r \int_{[a,a']} | A |(x)(x_1 - a_1)\, dx > 0.$$

This completes the proof of the lemma.

LEMMA 4.2. *There exists in R^k the real valued, nonnegative function $D(t)$ such that for each $t \in R^k$, $| \rho_b(t)| \leqslant D(t)$, independently of b.*

Proof. If any of the numbers $t_1,..., t_k$ are zeros, define $D(t) = 0$, for in this case each $\rho_b(t) = 0$. Let $t = (t,..., t_r) \in R^k$ be such that $t_r \neq 0$, $r = 1, 2,..., k$ and put $T(t) = \max\{| t_1 |,..., | t_k |\}$, $N(t) = (T(t),..., T(t)) \in R^k$. Then $T(t) > 0$ and so by the previous lemma we may choose $a_r'(t) > a_r$,

$r = 1,\ldots, k$ and $\epsilon(t) > 0$ so that $\int_{[a,a']} |A|(x)\, \varphi(x, s)\, dx \geqslant \epsilon$ for $|s_r| \leqslant T$, $r = 1,\ldots, k$. Then the Parseval equality for the function $\chi_{[a,a']}(x)$ yields

$$\begin{aligned}\int_{[a,a']} |A|(x)\, dx &= \int_{R^k} \left| \int_{[a,a']} |A|(x)\, \varphi(x, s)\, dx \right|^2 d\rho_b(s) \\ &\geqslant \int_{[-N,N]} \left| \int_{[a,a']} |A|(x)\, \varphi(x, s)\, dx \right|^2 d\rho_b(s) \\ &\geqslant \epsilon^2 \Delta_{\rho_b}([0, t]) = \epsilon^2 |\rho_b(t)|\end{aligned}$$

since $\rho_b(t) = 0$ if any of the numbers $t_1 ,\ldots, t_k$ are zero. Here, ϵ, a', T and N represent $\epsilon(t)$, $a'(t)$, $T(t)$ and $N(t)$. Note that $[0, x]$ is, strictly speaking, only defined if each $t_r \geqslant 0$. If, for example, $t_1 \leqslant 0$, $t_r \geqslant 0$, $r = 2,\ldots, k$, we interpret $[0, t]$ to mean $\{(s_1 ,\ldots, s_k) \mid t_1 \leqslant s_1 \leqslant 0, 0 \leqslant s_r \leqslant t_r , r = 2,\ldots, k\}$.

We now define $D(t)$ by

$$D(t) = \frac{1}{\epsilon^2(t)} \int_{[a,a'(t)]} |A|(x)\, dx.$$

Note also that this result is independent of the choice of $\beta_1 ,\ldots, \beta_k$ for the boundary conditions at b.

We now vary the point $b = (b_1 ,\ldots, b_k)$ by making some or all of the $b_r \to \infty$ through a sequence b_r^m say. This will give a sequence ρ_m of positively monotonic functions on R^k.

THEOREM 4.1. *There is a subsequence of ρ_m converging pointwise to a limiting spectral function ρ.*

Proof. For a positive integer n, let $N_n = (n, n,\ldots, n) \in R^k$. It follows from Lemma 4.2 that

$$V(\rho_m , [-N_n , N_n]) \leqslant 2^k D(N_n).$$

Thus by the Helley Theorem we may select a subsequence ρ_{1m} of ρ_m converging to a limit on $[-N_1 , N_1]$. Repeating the process we select a subsequence ρ_{2m} of ρ_{1m} converging to a limit on $[-N_2 , N_2]$ (and hence on $[-N_1 , N_1]$). Generally, we obtain a subsequence ρ_{nm} of $\rho_{(n-1)m}$ converging on $[-N_n , N_n]$. Then the subsequence $\rho_{m_p} = \rho_{pp}$ of ρ_m converges to a limit ρ at each point of R^k. ρ is positively monotonic and $\rho(t_1 ,\ldots, t_k) = 0$ if any of the numbers $t_1 ,\ldots, t_k$ are zero.

5. The Limiting Parseval Equality

We concern ourselves now with the Hilbert space H of functions $f(x)$ such that

$$\int_{[a,b]} |A|(x)\,|f(x)|^2\,dx < \infty$$

where $-\infty < a_r < b_r \leqslant \infty$, $r = 1, 2,..., k$.

Lemma 5.1. *Let $f(x) = \prod_{r=1}^{k} f_r(x_r) \in H$ be such that each f_r has absolutely continuous first derivative, $f_r'' \in L^2(a_r, b_r)$ and f_r vanishes in a neighborhood of $x_r = a_r$ and in a neighborhood of $x_r = b_r$ (or for x_r sufficiently great if $b_r = \infty$). Define*

$$F(t) = \int_{[a,b]} |A|(x)\,f(x)\,\varphi(x,t)\,dx.$$

Then

$$\int_{[a,b]} |A|(x)\,|f(x)|^2\,dx = \int_{R^k} |F(t)|^2\,d\rho(t).$$

Proof. For each $r = 1, 2,..., k$ we define

$$(T_r f)(x) = \Big(\prod_{s\neq r} f_s(x_s)\Big)\Big(\frac{d^2 f_r(x_r)}{dx_r^{\,2}} + q_r(x_r)\,f_r(x_r)\Big)$$

and $g^r(x)$ by the equations

$$(T_r f)(x) + \sum_{s=1}^{k} a_{rs}(x_r)\,g^s(x) = 0, \qquad r = 1, 2,..., k.$$

Note that these equations do define the g^r since we assume $|A|(x) > 0$.

Let us denote the inverse of the matrix $(a_{rs}(x_r))$ by $b_{rs}(x)/|A|(x)$. Now we have, for $p = 1, 2,..., k$,

$$\int_{[a,b]} |A|(x)\,g^p(x)\,\varphi(x,t)\,dx = -\int_{[a,b]} |A|(x)\,\frac{\sum_{s=1}^{k} b_{ps}(x)(T_s f)(x)}{|A|(x)}\,\varphi(x,t)\,dx$$

$$= -\sum_{s=1}^{k}\int_{[a,b]} b_{ps}(x)(T_s f)(x)\,\varphi(x,t)\,dx. \qquad (*)$$

We note that the function b_{ps} does not depend on x_s, so using $\varphi(x, t) = \prod_{r=1}^{k} \varphi_r(x_r, t)$, Fubini's Theorem and integration by parts, we see that the quantity (*) equals

$$\sum_{s=1}^{k} \int_{[a,b]} b_{ps}(x) f(x) \sum_{q=1}^{k} t_q a_{sq}(x_s) \varphi(x, t)\, dx$$

$$= \int_{[a,b]} \sum_{q=1}^{k} \left(\sum_{s=1}^{k} b_{ps}(x)\, a_{sq}(x_s) \right) t_q f(x)\, \varphi(x, t)\, dx$$

$$= \int_{[a,b]} |A|(x)\, t_p f(x)\, \varphi(x, t)\, dx.$$

Thus if we use the Parseval equality over a finite interval $[a, b^m]$ outside of which f vanishes, we have

$$\int_{[a,b]} |A|(x)\, |g^p(x)|^2\, dx = \int_{R^k} |t_p F(t)|^2\, d\rho_m(t).$$

Let $K > 0$ and denote by D the interval in R^k given by $|t_r| \leqslant K$, $r = 1,..., k$. Then if $t = (t_1, ..., t_k) \notin D$, there exists p, $1 \leqslant p \leqslant k$, such that $|t_p| > K$. Hence

$$|F(t)|^2 \leqslant \frac{|t_p F(t)|^2}{K^2} \leqslant \frac{1}{K^2} \sum_{p=1}^{k} |t_p F(t)|^2.$$

Now we have

$$\int_{[a,b]} |A|(x)\, |f(x)|^2\, dx = \int_{R^k} |F(t)|^2\, d\rho_m(t)$$

$$\leqslant \int_D |F(t)|^2\, d\rho_m(t) + \frac{1}{K^2} \int_{R^k} \sum_{p=1} |t_p F(t)|^2\, d\rho_m(t).$$

Thus

$$\left| \int_{[a,b]} |A|(x)\, |f(x)|^2\, dx - \int_D |F(t)|^2\, d\rho_m(t) \right|$$

$$\leqslant \frac{1}{K^2} \sum_{p=1}^{k} \int_{[a,b]} |A|(x)\, |g^p(x)|^2\, dx.$$

We now make $[a, b^m] \to [a, b]$ through the sequence which makes the spectral functions ρ_m converge and use Theorem 2.2. This yields the above inequality with ρ in place of ρ_m. Our result follows by letting $K \to \infty$.

THEOREM 5.1. *Let $f \in H$ be arbitrary. Then the integral*

$$\int_{[a,b]} |A|(x) f(x)\, \varphi(x, t)\, dx$$

converges in the mean in $L^2(\rho)$ *to a function* $F \in L^2(\rho)$ *such that*

$$\int_{[a,b]} |A|(x)\,|f(x)|^2\,dx = \int_{R^k} |F(t)|^2\,d\rho(t).$$

Proof. Firstly we consider the case where f vanishes outside the finite interval $[a, a'] \subset [a, b]$. Then there is a sequence $f^n(x)$ of finite linear combinations of functions of the form $\prod_{r=1}^{k} f_r(x_r) \in H$ where each f_r has absolutely continuous first derivative, $f_r'' \in L^2(a_r, b_r)$ and f_r vanishes in a neighborhood of $x_r = a_r$ and in a neighborhood of $x_r = a_r'$, and such that

$$\lim_{n\to\infty} \int_{[a,b]} |A|(x)\,|f(x) - f^n(x)|^2\,dx = 0.$$

Defining $F^n(t) = \int_{[a,b]} |A|(x) f^n(x)\,\varphi(x, t)\,dx$, we see from the previous lemma extended to finite linear combinations, that

$$\int_{R^k} |F^n(t) - F^m(t)|^2\,d\rho(t) = \int_{[a,b]} |A|(x)\,|f^n(x) - f^m(x)|^2\,dx$$
$$\to 0 \text{ as } n, m \to \infty.$$

Thus the sequence $F^n \in L^2(\rho)$ is Cauchy and so has a limit $F \in L^2(\rho)$. From the definition of F^n it is clear that

$$F(t) = \int_{[a,b]} |A|(x) f(x)\,\varphi(x, t)\,dx.$$

We now have

$$\begin{aligned}\int_{[a,b]} |A|(x)\,|f(x)|^2\,dx &= \lim_{n\to\infty} \int_{[a,b]} |A|(x)\,|f^n(x)|^2\,dx \\ &= \lim_{n\to\infty} \int_{R^k} |F^n(t)|^2\,d\rho(t) \\ &= \int_{R^k} |F(t)|^2\,d\rho(t),\end{aligned}$$

thus proving the theorem for this restricted class of functions.

If $f \in H$ is arbitrary and $[a, a'] \subset [a, b]$ is a finite interval, we define $f^{a'}(x) = f(x)\,\chi_{[a,a']}(x)$ and

$$F^{a'}(t) = \int_{[a,b]} |A|(x) f^{a'}(x)\,\varphi(x, t)\,dx.$$

Then from the above argument we see that if $a_r'' > a_r'$, $r = 1,\ldots, k$,

$$\int_{R^k} |F^{a'}(t) - F^{a''}(t)|^2\,d\rho(t) = \int_{[a,a'']-[a,a']} |A|(x)\,|f(x)|^2\,dx$$
$$\to 0 \text{ as } a', a'' \to b.$$

Thus the family $F^{a'}$ converges (as $a' \to b$) in the mean of $L^2(\rho)$ to a function $F \in L^2(\rho)$. Letting $a' \to b$ in

$$\int_{[a,a']} |A|(x) \cdot |f(x)|^2 \, dx = \int_{R^k} |F^{a'}(t)|^2 \, d\rho(t)$$

yields the desired result, i.e., the Parseval equality holds for arbitrary $f \in H$.

We conclude by giving an "expansion theorem" for arbitrary $f \in H$.

THEOREM 5.2. *Let $f \in H$ and $F \in L^2(\rho)$ its transform as defined in the previous theorem. Then the integral*

$$\int_{R^k} F(t) \, \varphi(x, t) \, d\rho(t)$$

converges in the mean in H to f.

Proof. It is clear from the system of Eqs. (1) that the functions $\varphi_r(x_r, t_1, ..., t_k)$, $r = 1, ..., k$ are real when $t_1, ..., t_k$ are real. Further, for $f, g \in H$ with transforms $F, G \in L^2(\rho)$, the identity

$$4f\bar{g} = |f + g|^2 - |f - g|^2 + i|f + ig|^2 - i|f - ig|^2$$

together with the Parseval equality gives

$$\int_{[a,b]} |A|(x) f(x) \overline{g(x)} \, dx = \int_{R^k} F(t) \overline{G(t)} \, d\rho(t).$$

We are given $f \in H$ with transform $F \in L^2(\rho)$. For $\Delta = [c, d)$ a finite subinterval of R^k, define

$$f_\Delta(x) = \int_\Delta F(t) \, \varphi(x, t) \, d\rho(t).$$

Let $g \in H$ with transform $G \in L^2(\rho)$ be such that $g(x)$ vanishes outside some finite subinterval $[a, a'] \subset [a, b]$. Then

$$\begin{aligned}
\int_{[a,b]} |A|(x) f_\Delta(x) \overline{g(x)} \, dx &= \int_{[a,a']} |A|(x) \overline{g(x)} \left(\int_\Delta F(t) \, \varphi(x, t) \, d\rho(t) \right) dx \\
&= \int_\Delta F(t) \left(\int_{[a,a']} |A|(x) \overline{g(x)} \, \varphi(x, t) \, dx \right) d\rho(t) \\
&= \int_\Delta F(t) \overline{G(t)} \, d\rho(t).
\end{aligned}$$

Thus if we put $\sim\Delta = R^k - \Delta$,

$$\left| \int_{[a,b]} |A|(x)(f(x) - f_\Delta(x)) \overline{g(x)}\, dx \right|^2 = \left| \int_{\sim\Delta} F(t) \overline{G(t)}\, d\rho(t) \right|^2$$

$$\leqslant \int_{\sim\Delta} |F(t)|^2\, d\rho(t) \int_{R^k} |G(t)|^2\, d\rho(t).$$

Now g is at our disposal, so let us take $g(x) = f(x) - f_\Delta(x)$ for $x \in [a, a']$, 0 otherwise. Then the inequality above gives

$$\int_{[a,a']} |A|(x)\, |f(x) - f_\Delta(x)|^2\, dx \leqslant \int_{\sim\Delta} |F(t)|^2\, d\rho(t).$$

Here the right hand side is independent of a', so letting $a' \to b$ we obtain the same inequality with $[a, a']$ replaced by $[a, b]$. The result follows by making $\Delta \to R^k$.

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